Generic derivative returns and carry measures for strategy development and testing

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Abstract

It is hard to find, in a single place, concise definitions of return calculations across different asset classes. In this note I will attempt to set out just this, by outlining how to calculate approximate excess returns for equity index futures, FX forwards, and interest rate swaps. My main focus is on liquid linear derivative products, instead of positions taken in cash-assets. As an addition, I will show how to derive simple common carry factors, which can (with some caveats) be used as a valid starting point for generating portfolio allocation signals.

1 Introduction

1.1 Returns – what, where, and how

Any prerequisite for portfolio allocations requires us to specify the returns we are targetting, which will be used in both back-tests and portfolio evaluations. Investors have a large array of asset classes available to them, all with different characteristics, and underlying pricing models. My focus in this note is on asset classes with positions taken in the liquid derivative space and for a daily frequency of returns.

Before we can discuss specific returns, an important distinction to make is that they can be measured in three different ways:

1. **Percent returns**: non-symmetric but aggregates on the cross section as a sum.

2. **Log returns**: symmetric and aggregates straightforwardly on the time-dimension as a sum.

3. **Per 100 USD invested at the outset**: symmetric dollar returns but with a rebalancing assumption built in.

The last one has the benefit of easily being able to take into account leverage of the portfolio and sums across both time and markets. However, it comes at the cost of having a rebalancing assumption built into it. In the calculations to follow, returns of equity index futures and FX forwards are in the percent measure. However, due to the log-linear approximation interest rate swap are log-returns.

Using a similar notation to Ang (2014) we define *gross returns* over one period (from $t-1$ to t) as

$$R_t = \frac{P_t + D_t}{P_{t-1}} \tag{1}$$

where R_t is the gross returns (in general $R_t \in [0, \infty]$), P_{t-1} is the assets price in the previous period (at time $t-1$, with $P_{t-1} \in [0, \infty]$ and $\Delta t = 1day$ given our daily frequency of returns)[1], D_t is the dividend paid out or interest earned in today (t), and P_t is today's price.[2] As can be seen from equation 1, the gross returns have two components to it; capital gains (P_t/P_{t-1}) and dividend yields (D_t/P_{t-1}). An implication of this is that a higher price today lowers yields on future dividends, assuming no change in the path of expected dividends.

From the gross returns in equation 1 the definition of *percent returns* follows straightforwardly as

$$r_t = 100(R_t - 1) \tag{2}$$

where r_t is the returns ($r_t \in [-100, \infty]$). They are not additive over time due to compounding of wealth, but are additive for the cross-section (across markets and assets), which makes them useful for portfolio construction. As an example the portfolio returns of r_t^p is given by the weighted sum of $r_t^p = \sum_i \omega_{t-1,i} r_{t,i}$ where $\omega_{t-1,i}$ is the weight of asset i in the portfolio and $r_{t,i}$ is the returns of asset i.

The symmetric *log (continuously compounded) returns* are defined as

$$\bar{r}_t = 100 \ln(R_t) \tag{3}$$

where $\bar{r}_t$ is the log returns ($\bar{r}_t \in [-\infty, \infty]$) and R_t is the gross returns as defined in equation 1. Log-returns aggregate easily across the time-dimension as a sum. However, they do not sum across securities and assets (a portfolio log return is not a simple weighted sum of the underlying log returns).

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†This note is dedicated to all my incredible colleagues and friends at Macrosynergy; past, present and future. I keep learning from and with you all – however, and needless to say, any errors or mistakes are all mine alone.

[1] One rare exception of negative prices was WTI oil futures on Monday the 20th of April 2020, when the front-future price turned negative due to the COVID-19 demand shock and limited storage capacity.

[2] We use the common convention that X_t refers to a discrete variable and $X(t)$ is a continuous one.

Lastly we can calculate the USD-returns of going long an asset ($r_t^{USD} \in [-(100 + \sum_{i=1}^{h} r_{t-i}^{USD}), \infty]$) as follows:

$$r_t^{USD} = \begin{cases} r_t & \forall t \in \{1, t_h, t_{2h}, \dots\} \\ \prod_{i=1}^{h}(1 + r_{t-i}/100) \times r_t \end{cases} \tag{4}$$

where we rebalance at 1, $t_h - 1$, $t_{2h} - 1$ and so forth. With USD returns, wealth aggregates straight forwardly as a sum.

The differences between the various return measures are illustrated in table 1. Log-returns are symmetric but only a first-order approximation of actual returns magnitude. However the use of percent returns would imply a higher volatility estimate as they are not symmetric (in magnitude). Instead it is worth considering logarithmic or USD returns for any volatility estimation (for single time-series of returns).

Table 1: Return calculations (illustrative)

Period	W_t	r_t	$\bar{r}_t$	r_t^{USD}
0	100			
1	50	−50%	−69%	-50
2	100	100%	69%	50
Avg.	83.33	25	0	0
S.D.	23.57	75	69	50

Where W_t is asset value in period t.

1.2 Futures and forward contracts

From Wikipedia we have the definition of a Futures (and forward) contract as a *"legal agreement to buy (long) or sell (short) something at a predetermined price at a specific time in the future, between parties not known to each other. The predetermined price the parties agree to buy and sell the asset for is known as the forward price. The specified time in the future – which is when delivery and payment occur – is known as the delivery date. Because it is a function of an underlying asset, a futures contract is a derivative product. "* Examples of assets commonly traded with listed futures contracts includes equity, commodities, developed markets government bonds, and volatility indices (such as VIX for the SP500). In contrast, although a futures market exists, FX forward contracts are usually traded as over-the-counter (OTC) directly between the two parties of the contract.

In the absence of arbitrage, convenience yields, and storage costs, the futures or forward price meets the following condition (Hull, 2018)

$$F_t^h = S_t(1 + i_t^h), \tag{5}$$

where F_t^h is the futures or forward price, S_t is the spot price, h is the time till maturity, and i_t^h is the risk-free rate for the period.

1.3 On Carry: Keep Calm

The concept of carry has a long-history in asset management, and various definitions exists. It is commonly summarised as *"an activity that provides a steady premium income, but exposes the seller to occasional large losses"*, colloquially known as picking pennies in front of a steamroller. As implied, carry can be interpreted as a steady stream of small insurance premiums paid against the risk of a large negative event (Sueppel, 2021). Our definition of the carry of a contract is the return over and above the risk-free rate that is earned if all prices in the market remain unchanged and all related economic flows, such as dividends, occur in line with expectations. For most macroeconomic carry measures Quantamental data, such as provided by the proprietary dataset JPMaQS, helps more accurately price this risk-premium, and access opportunities in this space, not captured by the pure price carry. This is an example of enhanced carry as described in Sueppel (2023).

In practical terms the definition we use for our calculations is the returns earned as time approaches maturity of the derivative contract, assuming no changes to the spot price ($S_{t+h} = S_t$, where h is maturity of the contract). One theory of carry is to follow Koijen et al. (2018, p.200) in defining the ex-post total return as

$$\begin{aligned} r_{t+1}^{tr} &= \frac{X_t(1 + i_t) + F_{t+1} - F_t - X_t}{X_t} \\ &= \frac{F_{t+1} - F_t}{X_t} + i_t \end{aligned} \tag{6}$$

where in the above r_{t+1}^{tr} is the *ex-post* total returns, F_t is price of a future or forward contract with expiry in period $t + 1$ (for now, we drop h for convenience), i_t is the risk-free interest rate and X_t is the capital allocated to the trade per contract bought (a scaling factor, which as minimum is equal to the margins posted per contract). From equation 6, the return in excess of the risk-free rate (the excess return) trivially follows as

$$r_{t+1}^{xr} = \frac{F_{t+1} - F_t}{X_t}, \tag{7}$$

where in the above r_{t+1}^{xr} is the ex-post excess returns at the end of the next period ($t + 1$).

Under the assumption of constant spot prices ($S_{t+1} = S_t$ which follows from an assumption of a random walk without drift) we have that $F_{t+1} = S_t$ as the future price is equal to the future spot ($F_{t+1} = S_{t+1}$). This implies a definition of carry as given by

$$C_t = \frac{S_t - F_t}{X_t}, \tag{8}$$

where C_t is the carry. As Koijen et al. (2018) says *"this definition makes it clear that carry is directly observable from current futures and spot prices. The Scaling factor X_t can be chosen freely depending on the needs of the researcher (or investor) as long as a consistent scaling of returns and carry is used."*. One interpretation, such as in Sueppel (2017), is to see the carry as the slope of the futures curve. If the period in question for the future (or forward) is not equal to a year, we can annualise the carry as $CRY_t = (1+C_t)^m$ where m is the annualisation factor (e.g. 12 for monthly, 52 if weekly and 252 when the contract is only for 1 business days).

2 Equities

2.1 A share

The term equity, in essence, refers to the purchase of stocks in a company which gives the holder a part in its ownership. A stock earns its owner dividend payments, and can be sold for capital gains (or loses) relative to its purchasing price. We interpret dividends, similar to Pedersen (2015, p.89), in being all cash paid to shareholders including share buybacks and net of any shareholder cash injections. They are in principle non-negative for limited liability companies, with the exceptions of shareholder cash injections. Dividends are discrete events in nature and conventionally paid out in fixed intervals such as annually or semi-annually. This implies that the daily time-series of dividends are zero, except for the (few) dates when they are paid out by the company. For this reason, it is conventional to use either a 12-month backward (or expected forward) looking window to get a measure of daily "annual dividends".

One approach to equity valuations is to calculate a stock's fundamental value (Pedersen, 2015, chapter 6), in terms of its discounted stream of expected payments to a holder.[3] This approach is known as the discounted dividend model, and is defined as

$$V_t = \mathbb{E}_t\left(\frac{D_{t+1} + V_{t+1}}{1+k_t}\right) \tag{9}$$

where V_t is the fundamental value (to make it distinct from the market value or price), D_{t+1} is dividends paid out (all net cash returned to shareholders), and k_t is the time-varying discount factor including any risk-premiums. Iterating forward equation 9, we get the net present value formulation of the valuation

$$\begin{aligned} V_t =& \mathbb{E}_t \sum_{j=t}^{T-1}\left(\frac{D_{t+j+1}}{\prod_{j=t}^{T-1}(1+k_{t+j})}\right) \\ &+ \mathbb{E}_t\left(\frac{V_{t+T}}{\prod_{j=t}^{T-1}(1+k_{t+j})}\right). \end{aligned} \tag{10}$$

The above has two terms, the net-present value of the dividend return streams, and the final "terminal value". It is common practise to assume for long-horizons (towards infinity in the limit) that the terminal value is zero.

For companies that go through growth phases a multi-stage dividend discount model is often used, as outlined in Pedersen (2015, p.91). This implies setting out a growth path (and discount factor) for the near future, and specifying a non-zero terminal value. For the last period value, Gordon's growth model is often used, as outlined in Pedersen (2015, p.90-91) or for the original articles outlining the idea see Gordon (1959, 1962) and Gordon and Shapiro (1956). This version of the dividend discount model assumes a constant expected growth rate for the dividend which means that $\mathbb{E}_T(D_{T+s}) = (1+g)^s D_T$, and so we write out the intrinsic value as

$$V_{t+T} = \frac{(1+g)D_{t+T}}{k-g}. \tag{11}$$

Lastly, combining with dividend discount model we get:

$$\begin{aligned} V_t =& \mathbb{E}_t \sum_{j=t}^{T-1}\left(\frac{D_{t+j+1}}{\prod_{j=t}^{T-1}(1+k_{t+j})}\right) \\ &+ \mathbb{E}_t\left(\frac{(1+g)D_{t+T}}{(k-g)\prod_{j=t}^{T-1}(1+k_{t+j})}\right). \end{aligned} \tag{12}$$

A variation of this model together with information from dividend futures and swaps is used in NBIM (2021) to estimate the term-structure of equity indices (and thereby their implied duration).

An alternative model of valuation, as outlined in Pedersen (2015), is to look at net earnings, dividend and book values in the residual income model to capture the impact of retained earnings for younger firms:

$$B_t = B_{t-1} + NI_t - D_t. \tag{13}$$

Here B_t is the book value, D_t is the dividend payout, and NI_t is net-income (profits). Solving for dividends we get what Pedersen (2015) refers to as the residual income model:

$$V_t = B_t + \sum_{s=1}^{\infty}\frac{\mathbb{E}_t(RI_t)}{(1+k)^s} \tag{14}$$

where RI is the residual income and defined as $RI_t = NI_t - kB_{t-1}$.

Retained earnings of a company, should add to it's assets and thereby increase the expected future dividend payments. This should result in prices adjusting today (from the expectations structure of dividends). See Petrella et al. (2020) for details on estimating trends in earnings and dividends, something also explored in NBIM (2021).

2.2 Equity futures contracts and excess returns

The equity index futures no-arbitrage equation from Koijen et al. (2018) is

$$F_t = S_t(1+i_t) - \mathbb{E}_t(D_{t+1}) \tag{15}$$

where $\mathbb{E}_t(D_{t+1})$ is the dividend (or earnings) expectations.

[3] The history of net present value calculations dates back to at least Fibonacci in the 13th century (Rubinstein, 2006, pp.3-5).

Given the fixed maturity dates of the equity future contracts, the excess returns are straight forwardly calculated as:

$$r_t^{xr} = 100\left(\frac{F_t^{h-1}}{F_{t-1}^{h}} - 1\right) \tag{16}$$

where F_t^{h-1} is the future contract with maturity $h-1$ (in trading days) today. The only caveat to the above is that on a roll-date, we sell (take profit) on the front future contract, and buy the second future contract with a later settlement date, which will become the next periods front future contract.

2.3 Carry calculations

As for the equity carry, following markets conventions it is calculated by first deriving the analysts expectations of the Earning Yields (12 month forward Earnings-Per-Share to price ratio):

$$\mathbb{E}_t(EY_{t+1}) = \frac{\mathbb{E}_t(EPS_{t+1y})}{S_t} \tag{17}$$

Where EPS_t is the Earnings-Per-Share, S_t is the spot price of the equity index, and EY_t is the Earnings yield. Similarly, we can calculate the Dividend yields as

$$\mathbb{E}_t(DY_{t+1}) = \frac{\mathbb{E}_t(DPS_{t+1y})}{S_t} \tag{18}$$

, where DPS_t is dividend-per-share. Company shares are typically real assets, and hence ratios to price results in real variables. Yet, funding costs are on nominal notional.

The carry for equity index futures are calculated the expected earning yield over the funding costs, in absence of any price changes.

$$CRY_t = \theta\mathbb{E}_t(EY_{t+1y}) + (1-\theta)\mathbb{E}_t(DY_{t+1y}) - i_t \tag{19}$$

where i_t is the funding cost (short-term interest rate), and we weight the carry between the earnings and dividend yield to take into account non-dividend payments such as retained earnings and share buybacks. In JPMaQS we use equal weights ($\theta = 0.5$ with $0 \leq \theta \leq 1$) between them. From this we can straightforwardly calculate the real equity carry as

$$CRR_t = CRY_t + \mathbb{E}_t(\pi_{t+1y}), \tag{20}$$

where the above holds due to the expected earnings yields ($\mathbb{E}_t(EY_{t+1y})$) already being a real variable, and hence we only need to adjust the nominal interest rate for inflation to get real carry.[4]

3 Currencies and foreign exchange rates

3.1 What is an exchange rate

A foreign exchange (FX) rate is the price at which we can sell one currency to purchase one unit of another. This is known as an FX cross (or equivalently a currency pair) and is given by the quote (or domestic) currency we are selling to purchase a unit of the base (or foreign) currency. An example of a cross is GBPUSD, commonly referred to as "cable", where GBP is the base and USD is the quote currency. A GBPUSD spot exchange rate of 1.225 implies that we can purchase 1 GBP for 1.225 USD by buying ("going long") the FX contract. By the nature of how a FX cross involves two currencies and how there is no natural numeraire between them (Clark, 2011, p.2), it will always be a relative trade of the economic conditions of the domestic (quote) against the foreign (base) currency's country.

3.2 Nominal appreciation

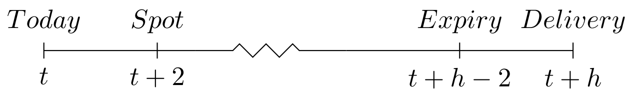
The nominal appreciation for a currency cross is calculated as

$$NA_t = \left(\frac{S_t}{S_{t-1}} - 1\right) \times 100 \tag{21}$$

where S_t is the spot exchange rate of a base against a quote currency. The spot rate (e.g. GBPUSD) is the price of selling S_t units of the quote (USD) for a unit of the base currency (GBP), and hence the nominal appreciation (NA_t) gives the increase in value (or decrease if negative i.e. depreciation) of the base relative to the quote currency.

3.3 Forward contracts

Figure 1: Horizon Dates



An FX forward contract commits its parties to exchange one currency for another at a pre-determined price and at a specific date in the future, say $t+h$. The date when settlement takes place is called the *delivery date*. For completeness, a *FX option* has the same *delivery date* as an FX forward contract, together with an expiry date for when the option holder must decide whether to exercise. The time-line of dates is commonly called the horizon dates and is illustrated in figure 1. For both settlement and delivery this implies transfer of funds e.g. selling the base currency and buying (getting) the quote currency. FX Broken Date: Forward Rates given a forward contract, with the definition of either (1) foreign exchange with unusual *delivery date*, such as a forward contract of USD to buy in 54 days (e.g. 3 months). (2) An unusual *value date*: a value date is the delivery date in a Euro currency or foreign exchange transaction and is usually a standard period such as 1 week or 1 month (3m, 6m, 9m, 12m – calendar month or from day to day, i.e. day count conventions). A broken date then may be a value date 13-days after the trade date. Value dates: dates when FX trades settles (payment of each currency is

[4] The basic results then follows that $CRR_t = \theta\mathbb{E}_t(EY_{t+1y}) + (1-\theta)\mathbb{E}_t(DY_{t+1y}) - (i_t - \mathbb{E}_t(\pi_{t+1y}))$.

made). Spot transactions are usual T+2 (e.g. two business days) with the exception of USDCAD which has only one business day delay for settlement. Value dates roll forward by 1-day at 5PM New York time by global convention.

3.4 Carry

The key no-arbitrage pricing condition for FX forwards is the *Covered Interest rate Parity* (CIP) theorem. The assumptions behind this is open markets and limited capital controls. It is derived from a zero-profit (no-arbitrage) condition of the following trade of going long the base currency (the foreign currency) by first borrowing S_t units of the domestic currency at the local interest rate (i_t^{dc}), which we then use to buy one unit of the foreign currency and deposit it earning the foreign interest rate (i_t^{fc}). Lastly in the first period we buy a forward contract to lock in the spot price to pay ($F_t = F_t(fc/dc)$). In the second period ($t+1$) the profits earned of going long the foreign currency is then given by $F_t(1+i_t^{fc}) - S_t(1+i_t^{dc})$. The income earned from the trade (and converted back to the domestic currency at the forward rate F_t) is $F_t(1+i_t^{fc})$, and the cost of borrowing is given by $S_t(1+i_t^{dc})$. If the zero profit holds it implies the covered interest parity no-arbitrage condition[5] for pricing the forward rate as in Koijen et al. (2018) and Sueppel (2017), adjusted for varying length of horizons for the forward contracts (using h) of

$$F_t = S_t \left(\frac{1+i_t^{dc}}{1+i_t^{fc}} \right)^h \tag{22}$$

where i_t^{dc} is the domestic (nominal) risk-free interest rate of the quote currency, i_t^{fc} is the foreign risk-free interest rate of the base currency, S_t is the spot rate (where the inverse e.g. $1/S_t$ is the cost of buying the base currency), and F_t is the forward rate with maturity h (which is usually a 1-month contracts implying $h = 1/12$). Both i_t^{fc} and i_t^{dc} are annualised rates, and hence the need to rescaling using h.

Using the above no-arbitrage equation 22 with the generic carry expression in equation 8 and with an allocation of 1 contract (e.g. $X_t = F_t$) and re-scaled to get annualised rates (using a factor of $1/h$), we calculate the annual FX carry (CRY_t) as

$$CRY_t = 100 \left(\left(\frac{S_t}{F_t} \right)^{1/h} - 1 \right). \tag{23}$$

This is equivalent to the formulation from Koijen et al. (2018) who see the FX trade as using the domestic currency to fund going long the foreign one, where the quote (funding) currency is the domestic and the base is the foreign.

[5]The assumption is $F_t(1+i_t^{fc}) - S_t(1+i_t^{dc}) = 0$. This is an assumption which has been questioned by some authors such as Borio et al. (2018). Partially this can be explained by the risk-free rate quoted not being without risk, especially in periods of financial stress.

Adjusting for the differential in inflation expectations between the two currencies we get the real carry as

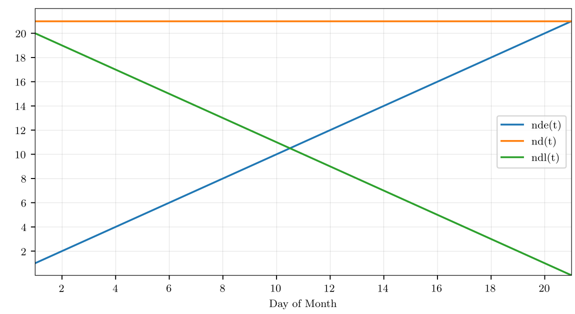
$$CRR_t = CRY_t - \mathbb{E}_t(\pi_{t+1}^{fc} - \pi_{t+1}^{dc}) \tag{24}$$

where CRR_t is the annual real percent carry, $\mathbb{E}_t(\pi_{t+1}^{fc})$ is the inflation expectations of the base (foreign) currency and $\mathbb{E}_t(\pi_{t+1}^{fc})$ is the inflation expectations of the quote (domestic) currency.

3.5 Excess returns

We calculate the returns of going *long the base currency* as follows. Due to the FX forward price data having fixed tenors, we need to estimate the roll-down of maturity for the contract held and how it impacts its price. This implies the excess returns are defined as the changes in the value of the forward contract, corrected for then broken dates due to the roll-down of maturity. To make this correction, we first calculate the convenience variables of the current business days in the month (nde_t), total number of business days in the month (nd_t), and the difference in business days till the end of the month (ndl_t). All three are illustrated in figure 2.

Figure 2: Day Count Fraction



Example for a month with 21 days ($k = 21$).

The first variable (nde_t) is calculated using the following algorithm:

$$nde_t = \begin{cases} 1 & \forall day(t) < day(t-1) \\ nde(t-1)+1 & \forall day(t) \geq day(t-1) \end{cases} \tag{25}$$

repeated for $t \in \{2, \ldots, T\}$, and where $day(t)$ is a function that returns the day of the month for date t. We start the iteration the prior month to the first observation in our sample.

The second variable (nd_t) is then defined as:

$$nd(t) = \begin{cases} nd(t+1) & \forall nde(t) < nde(t+1) \\ nde(t) & \forall nde(t) \geq nde(t+1) \end{cases} \tag{26}$$

for $t \in \{T-1, \ldots, 1\}$, with the initial condition that $nd_T = nde_T$. Where the updating only takes place for the case of the turn of the month.

Lastly we calculate the variable ndl_t as follows ($ndl \geq 0$)

$$ndl_t = nd_t - nde_t \quad \forall t \in \{1, \ldots, T\} \tag{27}$$

where ndl moves down 1-step until end of month. We will use the day count fraction (dcf) to adjust for the broken dated forwards contracts calculated as follows:

$$dcf_t = \frac{ndl_t}{252} \tag{28}$$

the number 252 is the working day adjustment, which follows a decay pattern as the month approaches the end.

The basic for excess returns calculations is to derive the FX forward (F_t) from the spot price and carry measure. Usually the data we observe is the price of buying a forward at a specific time, whereas we wouldn't observe the equivalent price of selling it (as the maturity would have changed). We calculate the forward broken dated contract (Fb_t) as follows from the covered interest parity and the carry formulation from equation 23 re-arranged to express as "synthetic" forward price adjusted for the remaining maturity

$$Fb_t = S_t \left(1 + CRY_t\right)^{-dcf_t} \tag{29}$$

where CRY_t is in units of annualised returns, and Fb_t is the fixed maturity rebalancing at the end of the month (synthetic forward). The (implicit) assumption is of buying a one-month contract at the start of the month and then holding on to it, until rebalancing at the end of the month. Here $day(t) = 1$ implies the start of a new month, and F_t is the price of a forward outright (1-month) contract. From the definition of the variables we have the $S_t \in \mathbb{R}^+$, $CRY_t \in \mathbb{R}$, and $dcf \in [0, 23/252]$ (bounded by the maximum amount of business/trading days in a month).

We can then derive the excess returns (r_t^{xr}) using the broken date forwards correction (to adjust for the differences in maturity) as

$$r_t^{xr} = \begin{cases} 100\left(Fb_t/F_{t-1} - 1\right) & \forall day(t) < day(t-1) \\ 100\left(Fb_t/Fb_{t-1} - 1\right) & \forall day(t) \geq day(t-1) \end{cases} \tag{30}$$

The forward and carry is calculated as in the previous section. Carry, with the deliverable forward contracts using spot exchange rate plus forward points and the non-deliverable forwards, using the (non-deliverable) forward rate directly. The underlying assumption of the excess returns is taking daily profits as the contracts matures (mark-to-market).

4 Interest rate swaps

4.1 Swapping payment streams

In it's simplest form a plain vanilla swap contract is a generic term for an OTC derivative in which two parties exchange a stream of cash flows against another, but without an exchange of the underlying notional (Andersen and Piterbarg, 2010, p.197). A common form of swap contracts is the fixed against floating. Examples of such contracts are the IRS often based on an interbank rate (such as LIBOR) or the Overnight Index Swaps (OIS) where the floating rate is what is commonly referred to as a Risk-Free Rate (RFR) such as the Effective Federal Funds rate (US) or Sonia (UK). Historically the floating leg of interest rate swaps was typically been an interbank-offered rate, such as LIBOR. However, after the rise in interbank credit risk during the great financial crisis and irregularities in data collection other short-term rates have gained importance.

In essence to price a swap contract ($V_{t,h}^{swap}$) with tenor h, you can set it out as pricing two different bond positions. One which you take a long position in (the fixed leg for a receiver position: $V_{t,h}^{fix}$), and the other for the floating leg (V_t^{float}):

$$V_{t,h}^{swap} = V_{t,h}^{fix} - V_t^{float}. \tag{31}$$

At the onset of the contract has a zero value ($V_{t,h}^{swap} = 0$). The value of a swap is equal to the expected discounted value of its netted payments (or equivalent of the portfolio between the two positions). We can think of the value of the fixed leg as per Fabozzi et al. (2012)

$$V^{fix} = \sum_{i=1}^{N \times k} \left(\frac{c/k}{(1 + y_k/k)^i} \right) \tag{32}$$

where y is the yield, and c is the coupon rate (derived from the no-arbitrage). As Fabozzi (2012, p.91) demonstrates we then have

$$V^{swap} = c \left(\frac{1 - \frac{1}{(1+y)^N}}{y} \right) \tag{33}$$

which again when $c = y$ reduces to

$$V^{swap} = 1 - \frac{1}{(1+y)^N}. \tag{34}$$

The JPMaQS setup for swap returns (and carry) to be taken as a long-duration position and hence to receive the payment streams of the fixed rate and pay floating rate costs (a receiver position). This implies a duration play, as it is equivalent to a long dated asset against (or funded) using short-term interest rates. It gives exposure to returns are positive when monetary policy is easing as it lowers the path of the floating rates paying. Being long the long-dated rate, implies a play on interest rates falling (short term rates going down), as the long-term rate are fixed, the cost will go down and hence the value of the asset will go up (the benefit of the discounted flows).

The price (value or traded price for coupon bearing bond) is given by the discounted value of the cash-flows:

$$V_t = \sum_{j=1}^{n} \frac{CF_{t+j}}{(1+y)^j} \tag{35}$$

where V_t is the value of the bond, y is the required yield, and CF is the cash-flow (eg. the expression for V_t is the discounted cash-flow). Similarly a discount price ($P^{(n)}$) is given by

$$P^{(n)} = \frac{1}{(1 + y^{(n)})^n} \geq 0 \quad \forall y^{(n)} > -1 \tag{36}$$

where $P^{(n)}$ is the price of a period n-discounted (zero-coupon) bond. Where if the underlying zero-coupon yields is positive this implies that $0 < P^n < 1$, if the yield is zero then $P^n = 1$, and if the yield is negative $P^n > 1$. For an n-period bond (with fixed coupon, no-repayments and no optionality), we can write the price (value) as

$$V_t^n = \sum_{j=1}^{n} \frac{c}{(1+y)^j} + \frac{1}{(1+y)^n}, \tag{37}$$

where V_t is the value of the bond, c (positive or zero: $c \geq 0$) is the coupon rate, y is the required yield, and n ($n > 0$) is the periods until maturity of the bond (principal is set to 1).

4.2 Excess returns

We base our return calculation on a log-approximation as originally derived by Shiller et al. (1983) result in log-returns as per equation 3. Effectively, it is a modification seeing combining the returns of the portfolio of the two legs in the contract into a single expression, where we get

$$\begin{aligned} r_{t,h}^{xr} = \quad & (y_{t-1,h}^{fix} - i_{t-1})/252 \\ & + D_{t-1,h}^{mod}(y_{t-1,h}^{fix} - y_{t-1,h-1}^{fix}) \\ & + D_{t-1,h}^{mod}(y_{t-1,h-1}^{fix} - y_{t,h-1}^{fix}), \end{aligned} \tag{38}$$

here $y_{t,h}^{fix}$ is the fixed rate with tenor h, i_{t-1} is the floating rate, lastly $D_{t-1,h}^{mod}$ is the modified duration, which captures the sensitive of the contract to changes in the price. The first term captures the daily carry earned from the yield difference $((y_{t-1,h}^{fix} - i_{t-1})/252)$. The second term is the impact of a change in price (yields), and how it impacts the value of the contract measure adjusted for sensitivity by the modified duration measure. Two import points to note on our return measure is that JPMaQS don't usually include the curve roll down component by 1-trading day due to tenors only being available in fixed intervals, and secondly the floating rate resets on a daily frequency to the current rate.

The duration gives the maturity of a zero-coupon bond that will have the same price as the n period bond with cash-flows (coupon) payments. The Frederick Macaulay Duration is defined as the present value of cash flows which for periodically compounded yields is defined as[6]

$$V(y_k) = \sum_{i=1}^{n} PV_i = \sum_{i=1}^{n} \left(\frac{CF_i}{(1 + y_k/k)^{kt_i}} \right), \tag{39}$$

which implies that Macaulay duration of

$$MacD = \sum_{i=1}^{n} \frac{t_i}{V(y_k)} \frac{CF_i}{(1 + y_k/k)^{t_i k}} \tag{40}$$

[6]It was first outlined by Frederick Macaulay in his NBER book (Macaulay, 1938).

The *Macaulay duration* is weighted average time until repayment (units: years e.g. time). The modified duration with periodically compounding is then given by

$$\begin{aligned} \frac{\partial V}{\partial y_k} &= \frac{-1}{(1 + y_k/k)} \sum_{i=1}^{n} t_i \frac{CF_i}{(1 + y_k/k)^{kt_i}} \\ &= -\frac{MacDV(y_k)}{(1 + y_k/k)}. \end{aligned} \tag{41}$$

We can re-arrange the above expression to express the "established relationship"' between the modified and Macaulay duration as

$$ModD = \frac{MacD}{1 + y_k/k} = -\frac{1}{V(y_k)} \frac{\partial V}{\partial y_k}. \tag{42}$$

The *Modified duration* is a price sensitivity measure when the price is treated as a function of yield, and hence the modified duration can be seen as the semi-elasticity with respect to yields The units of modified duration is in terms of percentage change in price per one unit change in yield per year.

4.3 Carry

A swap contracts carry has two components to it of a difference in yield between the fixed and floating rates, and a roll-down of the curve with yields of the fixed rate converging to the lower tenor. As per our definition of carry, there is no direct price change impact, as the assumption is that all prices stays the same. Use these assumptions our carry calculations are

$$CRY_{t,\tau} = y_{t,h}^{fix} - i_t + D_{t,h}^{Mod}(y_{t,h-1y}^{fix} - y_{t,h}^{fix}) \tag{43}$$

where similar to the excess returns $y_{t,h}^{fix}$ is the fixed rate, i_t is the floating rate, and $D_{t,h}^{mod}$ is the modified duration as described above. The yield differential is the first term of $y_{t,h}^{fix} - i_t$, and the roll-down component is given by $D_{t,h}^{mod}(y_{t,h}^{fix} - y_{t,h-1}^{fix})$.

As both legs of the contract are equally affect by inflation the real carry is just equal to the nominal carry of:

$$CRR_{t,h} = CRY_{t,h}, \tag{44}$$

where $CRR_{t,h}$ is the real carry for for duration of tenor h, and $CRY_{t,h}$ is the equivalent nominal carry as defined above.

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